Topology and its Applications 33 (1989) 163-172
North-Holland

IMAGES OF THE CANTOR FAN

J.J. CHARATONIK and W.J. CHARATONIK
Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received 29 December 1987

Structural characterizations are obtained of images of the Cantor fan (i.e., the cone over the Cantor set) under mappings that belong to one of the following classes: confluent, open, monotone, retractions, light, and any intersections of these. A necessary and sufficient condition is shown under which there exists a monotone mapping from an arbitrary fan onto an arc.

AMS (MOS) Subj. Class.: 54C10, 54F50

image	end point	property of Kelley
light	monotone	Cantor fan
open	dendroid	continuous
arc	retract	confluent
fan	smooth	

A *continuum* means a compact connected metric space. It is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid.* A point x of a dendroid X is called an *end point* of X provided x is an end point of any arc in X that contains x. The set of all end points of a dendroid X is denoted by $E(X)$. A point x of a dendroid X is called a *ramification point* of X provided it is the vertex of a simple triod contained in X, i.e., if there are three arcs xa, xb and xc in X having x as the only point of the intersection of any two of them. By a *fan* we understand a dendroid having exactly one ramification point, usually denoted by v and called the *top* of the fan. Given a fan X, we put $S(X)=\{v\}\cup E(X)$. The cone F_C over the Cantor ternary set C is called the *Cantor fan.*

A dendroid X is said to be *smooth* provided there exists a point v in X such that for each sequence of points x_n of X converging to a point x the sequence of arcs vx_n converges to the arc vx [4, p. 298]. It is known that if a fan is smooth, then its top can be taken as a point v in the above definition [4, Corollary 9, p. 301].

All mappings considered in the paper are assumed to be continuous. A mapping $f: X\to Y$ between continua X and Y is said to be:

– *open,* if the image of any open subset of X is an open subset of $f(X)$;

– *monotone,* if for each subcontinuum Q of $f(X)$ the inverse image $f^{-1}(Q)$ is connected (equivalently: if $f^{-1}(y)$ is connected for each point y in Y);

– *monotone relative to a point* $x\in X$ if for each subcontinuum Q of $f(X)$ such that $f(x)$ is in Q, the inverse image $f^{-1}(Q)$ is connected;

– *confluent*, if for each subcontinuum Q of $f(X)$ and for each component K of $f^{-1}(Q)$ the equality $f(K)=Q$ holds;

– *light*, if for each point y of $f(X)$ the inverse image $f^{-1}(y)$ is zero-dimensional.

Thus monotone mappings are confluent; open ones are confluent too [13, Theorem 7.5, p. 148]. It is known that the nondegenerate image of a fan under a confluent mapping is either a fan or an arc; and if it is a fan, then the top of the domain is mapped onto the top of the range [2, Theorem 12, p. 32].

The following four propositions and a corollary will be useful in the sequel.

1. Proposition. *If a surjective mapping $f: X \to Y$ from a fan X with the top v onto a fan Y with the top v' is confluent, then for each point x of X the partial mapping $f|vx$ is monotone, and it maps the arc vx onto $v'f(x)$ (which may be degenerate).*

Proof. It has been proved in [2, Lemma 4, p. 32] that a confluent mapping between fans is monotone relative to the top of the domain. This is equivalent, by [11, Corollary 2.10, p. 722], to the condition stated in the conclusion. □

2. Proposition. *If a surjective mapping $f: X \to Y$ between fans X and Y is confluent, then $f(S(X)) = S(Y)$.*

Proof. Let a point y be in $f(S(X))$. Thus there is a point x in $S(X)$ with $y=f(x)$. If x is just the top v of X, then y is the top v' of Y [2, Theorem 12, p. 32], whence y is in $S(Y)$. So let x be in $E(X)$, and suppose on the contrary that $y \in Y \setminus S(Y)$. Take an end point e of Y with $y \in v'e$. Denote by K the component of $f^{-1}(ye)$ containing x, and observe that if v is in K, then $vx \subset K$, whence $v' \in f(vx) \subset f(K) = ye$, a contradiction. Thus K is a proper subarc of the arc vx, but then $f(K) \subset f(vx) \subset v'y$, a contradiction to confluence of f. So one inclusion is proved.

To see the other, let y be in $S(Y)$. If y is the top of Y, then $y=f(v) \in f(S(X))$, and we are done. If y is in $E(Y)$, take a point x in $f^{-1}(y)$. Thus $x \neq v$, and there is exactly one end point e of X such that $x \in ve$. By Proposition 1, the set $f(ve)$ is an arc having $f(v)$ and $f(e)$ as its end points. Since $f(x)=y$ is a point of this arc and y is in $E(Y)$, we see that $y=f(e)$ (moreover, $f(xe)=\{y\}$). Thus y is in $f(E(X)) \subset f(S(X))$. The proof is complete. □

3. Proposition. *Let a surjective open mapping $f: X \to Y$ be defined on a smooth fan X. If Y is a fan and if v and v' denote the tops of X and Y respectively, then*

$$f^{-1}(v') = \{v\} \quad \textit{and} \quad f(E(X)) = E(Y).$$

Proof. To prove the former equality suppose on the contrary that there is a point $x_0 \in X \setminus \{v\}$ such that $f(x_0) = v'$. Choose $e_0 \in E(X)$ with $x_0 \in ve_0$, take a point $e' \in E(Y) \setminus \{f(e_0)\}$ and put $K = \bigcup \{vx: x \in f^{-1}(e')\} \subset X$. We shall show that K is closed. In fact, let a sequence of points $a_n \in K$ converge to a point $a \in X$. Since a_n are in K, there is a sequence of points $b_n \in X$ with $a_n \in vb_n$ and $b_n \in f^{-1}(e')$. Choosing a convergent subsequence if necessary, we may assume that the points b_n converge to a point $b \in X$. Then $b \in f^{-1}(e')$ by continuity of f. Since X is smooth, the arcs vb_n converge to the arc vb, and since $a_n \in vb_n$, we infer that $a \in vb$, whence $a \in K$. Thus K is closed. Therefore $X \setminus K$ is an open subset of X, and so its image $f(X \setminus K)$ is open in Y. Note that if a point e is in $K \cap E(X)$, then $f(e) = e'$. Consequently, e_0 is out of K and so is x_0 by Proposition 1. Thereby $f(X \setminus K)$ is an open neighborhood of v' in Y. Hence there is a point y in $f(X \setminus K) \cap (v'e' \setminus \{v', e'\})$. Take a point $x \in X \setminus K$ with $f(x) = y$ and an end point e_1 of X with $x \in ve_1$. Thus $ve_1 \setminus \{v\} \subset X \setminus K$. In particular e_1 is not in K. Since $f(ve_1)$ is an arc from v' to an end point of Y by Propositions 1 and 2, and since this arc contains the point y, it is just the arc $v'e'$. So, by the definition of K we have $ve_1 \subset K$, a contradiction.

To prove the latter equality recall that $E(Y) \subset f(E(X))$ by Proposition 1 and that $f(E(X)) \subset S(Y) = \{v'\} \cup E(Y)$ by Proposition 2. Hence it is enough to show that there is no end point e of X with $f(e) = v'$. But this is a consequence of the former equality. Thus the proof is complete. □

4. Proposition. *A fan is smooth if and only if it can be embedded into the Cantor fan.*

Proof. One way is obvious since the Cantor fan is smooth and smoothness is a hereditary property [4, Corollary 6, p. 299]. To see the other way note that for each smooth fan X with the top v there is a mapping $f: X \to [0, 1]$ such that for each $x \in X$ the partial mapping $f|vx$ is one-to-one [5, Corollary 4, p. 90]. This mapping f is employed to construct the needed embedding in [2, Theorem 9, p. 27]. □

As an immediate consequence of Proposition 4 we have the following corollary.

5. Corollary. *Each smooth fan can be represented as the union of straight line segments in the plane.*

In the next result, i.e., in Theorem 6 below, confluent images of the Cantor fan are characterized in several ways. Condition (iv) of this theorem has been suggested to the authors by J.R. Prajs. To formulate the result we recall two definitions.

A subset Y of a space X is said to be a *retract* of X provided there exists a *retraction* f of X onto Y, that is, such a mapping that $f|Y$ is the identity.

A continuum X is said to have the *property of Kelley* provided that for each point x in X, for each sequence of points x_n converging to x and for each subcontinuum L of X containing x there exists a sequence of subcontinua L_n of X with x_n in L_n that has L as its limit [6, Property 3.2, p. 26].

6. Theorem. *The following conditions are equivalent for a nondegenerate continuum Y:*

(i) *Y is the image of the Cantor fan F_C under a confluent mapping;*

(ii) *Y is embeddable into the Cantor fan F_C, and for each (for some) embedding h of Y into F_C the image $h(Y)$ is a confluent retract of F_C;*

(iii) *Y is embeddable into the Cantor fan F_C, and for each (for some) embedding h of Y into F_C the image $h(Y)$ is a light retract of F_C;*

(iv) *Y is embeddable into the Cantor fan F_C, and for each (for some) embedding h of Y into F_C the image $h(Y)$ is a retract of F_C;*

(v) *Y is either a fan having the property of Kelley or an arc;*

(vi) *Y is either a smooth fan with the closed set $S(Y)$ or an arc.*

Proof. Two circles of implications will be shown: (vi)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(vi) and (vi)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi). In the former circle the implication (ii)$\Rightarrow$(i) is trivial. To see (i)$\Rightarrow$(vi) let a surjection $f: F_C \to Y$ be confluent. Then Y is either a smooth fan or an arc by [2, Theorem 13, p. 33]. If it is a fan, then $S(Y)$ is closed by Proposition 2.

(vi)$\Rightarrow$(ii). Consider any embedding h of Y into F_C. To simplify notation we omit the embedding in our considerations, and we assume that Y itself is a subset of F_C. To show that Y is a confluent retract of F_C denote by v the top of F_C and put

$$E = \mathrm{cl}\{e \in E(F_C) \colon Y \cap ve \neq \{v\}\}.$$

Since each closed subset of the Cantor set $C = E(F_C)$ is its retract [8, Section 26 II, Corollary 2, p. 281], there exists a retraction $r: E(F_C) \to E$. Let $f: F_C \to f(F_C) \subset F_C$ be the linear extension of r. Observe that f is confluent and the partial mapping $f|Y$ is the identity. Define $s: f(F_C) \to Y$ as follows: $s|Y$ is the identity; and for the closure K of any component of $f(F_C) \setminus Y$, the image $s(K)$ is the singleton $K \cap Y$. So s is well defined, and since $S(Y)$ is closed, s is continuous. Note that s is a monotone retraction. Since the composition of two confluent mappings is confluent [1, III, p. 214], we see that $sf: F_C \to Y$ is the needed confluent retraction.

In the latter circle of implications, the one from (iii) to (iv) is trivial. To see that (iv) (the version "for some") implies (v) note that F_C has the property of Kelley, which is an invariant under retractions [12, Theorem 2.9, p. 294]. Fans Y having the property of Kelley are characterized [3, Theorem 3] as smooth ones with the closed set $S(Y)$. Thus the implication (v)$\Rightarrow$(vi) holds true, and we have only to show that (vi) implies (iii).

(vi)$\Rightarrow$(iii). Exactly like in the first half of the proof of the implication (vi)$\Rightarrow$(ii), we assume that Y is a subset of F_C, we define the same closed subset E of $E(F_C)$, a retraction $r: E(F_C) \to E$ and the linear extension $f: F_C \to f(F_C) \subset F_C$ of r. Again the partial mapping $f|Y$ is the identity, and note that f is light.

Consider first the case when Y is an arc. Thus the set E consists of one or two points, and we see that $f(F_C)$ is an arc too. Then, if $Y = f(F_C)$, the proof is finished; if $f(F_C) \setminus Y$ is not empty, it consists of at most two components. Define a retraction $s: f(F_C) \to Y$ taking $s|Y$ as the identity, and for the closure K of a component of $f(F_C) \setminus Y$ putting $s|K : K \to Y$ as a homeomorphism being the identity on the singleton $K \cap Y$. Then $sf: F_C \to Y$ is the needed light retraction.

Consider now the case when Y is a fan. Note that for any component of $f(F_C) \setminus Y$ its closure K is a straight line segment having one end point, $y(K)$, in Y and the other, $e(K)$, in $E(f(F_C)) \subset E(F_C)$. Take an auxiliary straight line segment va of length equal to diam F_C having only the top v of F_C in common with F_C. Define a (not necessarily surjective) mapping $s: f(F_C) \to Y \cup va$ taking again $s|Y$ as the identity and, for the closure K of any component of $f(F_C) \setminus Y$, understanding the partial mapping $s|K$ as an isometry of $K = y(K)e(K)$ into the union of two straight line segments $y(K)v \cup va$ with $s(y(K)) = y(K)$. Thus s is well defined and, since $S(Y)$ is closed, it is continuous. Moreover, by the definition, s is light. Finally, retract $sf(F_C)$ onto Y under a mapping g that is the identity on Y and that homeomorphically maps the intersection $va \cap sf(F_C)$, if nondegenerate, onto a fixed arc ve, where $e \in E(Y)$. Again g is light, and therefore the composition $gsf: F_C \to Y$ is the needed retraction. The proof is complete. □

To show the next result we need a lemma.

7. Lemma. *For each metrizable compact zero-dimensional set A there is an embedding g of A into the Cantor set C such that the image $g(A)$ is an open retract of C.*

Proof. Let A be such a set. Thus the product $A \times C$ is homeomorphic with C (as a compact zero-dimensional set without any isolated points). Its natural projection onto A is an open retraction. □

8. Theorem. *The following conditions are equivalent for a nondegenerate continuum Y:*

(i) *Y is the image of the Cantor fan under an open mapping;*

(ii) *Y is the image of the Cantor fan under a light open mapping;*

(iii) *Y is the image of the Cantor fan under a light confluent mapping;*

(iv) *there exists an embedding h of Y into the Cantor fan F_C such that the image $h(Y)$ is an open retract of F_C;*

(v) *there exists an embedding h of Y into the Cantor fan F_C such that the image $h(Y)$ is a light open retract of F_C;*

(vi) *there exists an embedding h of Y into the Cantor fan F_C such that the image $h(Y)$ is a light confluent retract of F_C;*

(vii) *Y is either a smooth fan with the closed set of its end points, or an arc.*

Proof. Note that it is enough to establish the equivalence of all conditions except (iv) and (vi), because the implications (v)⇒(iv)⇒(i) and (v)⇒(vi)⇒(iii) are evident. Further, since the implications (v)⇒(ii), (ii)⇒(i) and (ii)⇒(iii) also are obvious, to complete two circles of implications, namely (vii)⇒(v)⇒(ii)⇒(i)⇒(vii) and (vii)⇒(v)⇒(ii)⇒(iii)⇒(vii) (which suffice to establish the equivalence of conditions (i), (ii), (iii), (v) and (vii)), we have to show three implications only, viz. (i)⇒(vii), (vii)⇒(v) and (iii)⇒(vii).

(i)⇒(vii). Let a surjection $f: F_C \to Y$ be open. Since open mappings are confluent, Theorem 5 can be applied, and thereby Y is either a smooth fan with the closed set $S(Y)$ or an arc. So we have only to show that, in case Y is a fan, the set $E(Y)$ is closed. But since $E(F_C)$ is closed, this is a consequence of Proposition 3.

(vii)⇒(v). Since Y is either a smooth fan or an arc, it can be represented, according to Corollary 5, as the union of straight line segments. Let v' denote the top of Y in case it is a fan, or an interior point of Y otherwise. In the latter case Y is assumed to be the union of two straight line segments, each joining v' with an end point of Y. We construct the needed embedding h of Y into the Cantor fan F_C.

Since $E(Y)$ does not contain any nondegenerate continuum [10, 2.1, p. 302] and is closed, thus compact, it is zero-dimensional. According to Lemma 7 there is an embedding $g: E(Y) \to g(E(Y)) \subset E(F_C)$ such that $g(E(Y))$ is an open retract of $E(F_C)$, i.e., that there exists an open retraction $r: E(F_C) \to g(E(Y))$. Denote by $h: Y \to h(Y) \subset F_C$ the linear extension of g. Since $E(Y)$ is closed, h is continuous. Thus, being one-to-one by its definition, it is an embedding. Now we linearly extend the open retraction r to $f: F_C \to h(Y)$. Continuity of f is obvious. Since r is a retraction of $E(F_C)$ onto $r(E(F_C)) = g(E(Y)) = h(E(Y))$, and since h and f are linear extensions of g and r respectively, we see that f is a retraction from the cone F_C over the domain $E(F_C)$ of r onto the cone $f(F_C) = h(Y)$ over the range $r(E(F_C)) = h(E(Y))$ of r. Further, for each point y in Y the set $f^{-1}(h(y))$ is compact and contains no nondegenerate subcontinuum, whence it is zero-dimensional. Thus f is light. Openness of f is a consequence of that of r and of linearity of the partial mappings $f|ve$ for all $e \in E(F_C)$, where v means the top of F_C.

(iii)$\Rightarrow$(vii). Let a surjection $f: F_C \to Y$ be light confluent. Thus Y is either a smooth fan or an arc by Theorem 6. If Y is a fan, let v' denote its top. Suppose on the contrary that $E(Y)$ is not closed. Therefore by Proposition 2 there is a point $e \in E(F_C)$ with $f(e) = v'$; but then $f(ve) = \{v'\}$ according to Proposition 1, and so f is not light. The proof is finished. □

Note that we cannot replace the phrase "there exists an embedding" in conditions (iv), (v) and (vi) of Theorem 8 by "for each embedding" (as it was done in the formulation of conditions (ii), (iii) and (iv) of Theorem 6). Namely if $Y \subset F_C$ is a homeomorphic copy of F_C diminished twice with respect to the normal size of F_C (i.e., Y is the image of F_C under a homothetic transformation of ratio $\frac{1}{2}$ and of centre at the top of F_C), then each confluent retraction of F_C onto Y shrinks each component of $F_C \setminus Y$ to the corresponding end point of Y lying in the closure of the component. Actually, such a confluent retraction is monotone relative to the top of F_C by [2, Lemma 4, p. 32].

Let a fan X be given. An arc $ab \subset X$ is said to be *free* provided $ab \setminus \{a, b\}$ is an open subset of X. We have the following result.

9. Theorem. *A fan X can be mapped onto an arc under a monotone mapping if and only if X contains a free arc.*

Proof. If a fan X contains a free arc ab, then $X \setminus (ab \setminus \{a, b\})$ has exactly two components: A, to which the point a belongs, and B, that contains b. Define a surjection $f: X \to ab$ putting $f(x) = x$ if $x \in ab$, $f(x) = a$ if $x \in A$ and $f(x) = b$ if $x \in B$. Thus f is a monotone retraction.

Now let f be a monotone surjection from X onto an arc ab, and suppose on the contrary that there is no free arc in X. Since $f^{-1}(a)$ and $f^{-1}(b)$ are two disjoint subcontinua of X, at most one of them contains the top v. So, let v be out of $f^{-1}(a)$. Thus $f^{-1}(a)$ is an arc or a point contained in an arc ve_0 for some end point e_0 of X. Denote by d a metric on X and assume (without loss of generality) that the metric d' on the arc ab is convex. Take a positive number $\varepsilon < d'(a, f(v))$ and let a positive number $\delta < d(v, f^{-1}(a))$ be chosen such that if $d(x_1, x_2) < \delta$, then $d'(f(x_1), f(x_2)) < \varepsilon$ for all points x_1, x_2 in X. Pick a point x_0 in $f^{-1}(a)$. Since no arc is free in X, the δ-ball about x_0 contains a point x that lies in an arc ve for some $e \in E(X) \setminus \{e_0\}$. Hence the arc xx_0 contains the top v. Further, the partial mapping $f|xx_0$ is monotone [4, Proposition 1, p. 307], but obviously not constant, and therefore $f(xx_0)$ is an arc from $f(x_0)$ to $f(x)$ [9, Section 48 I, Theorem 3, p. 192] with $d'(a, f(x)) < \varepsilon$. By convexity of d' we conclude that the whole arc $af(x) = f(xx_0)$ is contained in the ε-ball about a, whence $d'(a, f(v)) < \varepsilon$, a contradiction. The proof is complete. □

For each $n \in \{1, 2, 3, \dots\}$ let F_C^n denote a copy of the Cantor fan F_C with $\operatorname{diam}(F_C^n) < 1/n$, and let F_C^ω be the one-point union of all fans F_C^n with their tops identified. In other words, F_C^ω is homeomorphic to F_C / ve for some fixed $e \in E(F_C)$.

10. Theorem. *The following conditions are equivalent for a nondegenerate continuum Y:*

(i) *Y is the image of the Cantor fan under a monotone mapping;*

(ii) *there exists an embedding h of Y into the Cantor fan F_C such that the image $h(Y)$ is a monotone retract of F_C;*

(iii) *Y is homeomorphic either to F_C or to F_C^ω.*

Proof. Since the implication (ii)⇒(i) is trivial, to complete a circle of implications we need to show one from (i) to (iii) and one from (iii) to (ii).

(i)⇒(iii). Assume a surjection $f: F_C \to Y$ is monotone. Thus again Y is either a smooth fan or an arc, by Theorem 6. However, by Theorem 9, it cannot be an arc. So Y is a smooth fan, and we may assume, according to Proposition 4, that it is embedded into the Cantor fan. Denote by v the top of F_C (the domain of f), put $v' = f(v)$ and let $T = S(F_C) \setminus f^{-1}(v')$. Consider two cases.

(1) $v' \in Y \setminus \operatorname{cl} E(Y)$. Thus the singleton $\{v'\}$ is both closed and open in $S(Y)$. Since the partial mapping $f|S(F_C): S(F_C) \to S(Y)$ is a continuous surjection by Proposition 2, the intersection $S(F_C) \cap f^{-1}(v')$ is a closed and open subset of $S(F_C)$. Thus T is a closed and open subset of $E(F_C)$, so it is homeomorphic to the Cantor set. We prove that the partial mapping $f|T$ is a homeomorphism. To this end it is enough to show that $f|T$ is one-to-one. If not, there are two distinct points e_1 and e_2 in $E(F_C)$ such that $f(e_1) = f(e_2)$. Then $f^{-1}(f(e_1))$ is connected, so it contains v, a contradiction. Therefore we see that $E(Y) = f(T)$ is homeomorphic to the Cantor set, whence we conclude that the cones over these sets also are homeomorphic, i.e., that Y is homeomorphic to F_C.

(2) $v' \in \text{cl}\, E(Y)$. Then the set T is an open but not closed subset of $E(F_C)$. So it can be represented as the union of countably many elements of the standard base of the Cantor set $E(F_C)$, i.e., $T = \bigcup\{E_n: n \in \{1, 2, 3, \ldots\}\}$, where E_n are mutually disjoint, closed and open in $E(F_C)$ copies of the Cantor set, of diameters tending to zero if n tends to infinity. As in the previous case, one can show that for each n the partial mapping $f|E_n$ is a homeomorphism. Obviously $E(Y) = f(T) = \bigcup\{f(E_n): n \in \{1, 2, 3, \ldots\}\}$ and $\{v'\} = \text{Lim}\, f(E_n)$. Let F_n be the cone over $f(E_n)$ with the vertex v'. Observe that F_n is homeomorphic to the Cantor fan, $\text{Lim}\, F_n = \{v'\}$, $F_n \cap F_m = \{v'\}$ for $n \neq m$, and $Y = \bigcup\{F_n: n \in \{1, 2, 3, \ldots\}\}$. Thus one can define a natural homeomorphism between Y and F_C^ω.

(iii)$\Rightarrow$(ii). If Y is homeomorphic to F_C, then a given homeomorphism is the needed embedding. So let Y be homeomorphic to F_C^ω, i.e., $Y = \bigcup\{Y_n: n \in \{1, 2, 3, \ldots\}\}$, each Y_n being a copy of the Cantor fan, with $\text{Lim}\, Y_n = \{v'\} = Y_m \cap Y_n$ if $m \neq n$, where v' is the top of Y. For each $n \in \{1, 2, 3, \ldots\}$ put $E_n = [2/3^n, 1/3^{n-1}] \cap E(F_C)$. So each E_n is a portion of the Cantor set $E(F_C)$, and therefore

$$F_C^n = (E_n \times [1 - 1/n, 1]) / E_n \times \{1\}$$

can be considered as a subfan of the Cantor fan $F_C = (E(F_C) \times [0, 1]) / E(F_C) \times \{1\}$. Take homeomorphisms $h_n: Y_n \to F_C^n$ and define $h: Y = \bigcup Y_n \to \bigcup F_C^n \subset F_C$ putting $h|Y_n = h_n$ for each n. Then h is a homeomorphism that embeds Y into F_C. Now the monotone retraction from F_C onto $\bigcup F_C^n = h(Y)$ we need can be defined in a very natural way: each component K of $F_C \setminus h(Y)$ is mapped onto the singleton $h(Y) \cap \text{cl}\, K$. The proof is complete. □

Note that we cannot replace the phrase "there exists an embedding" in condition (ii) of Theorem 10 by "for each embedding", because if we take as Y a fan homeomorphic to F_C and embedded into F_C as its nowhere dense subset, then Y is not a monotone retract of F_C.

Theorems 8 and 10 imply that a nondegenerate continuum is the image of the Cantor fan under a monotone open mapping if and only if it is homeomorphic to the Cantor fan. However, a stronger result can be shown saying that the mapping under consideration is a homeomorphism itself.

11. Theorem. *Each monotone open mapping defined on the Cantor fan is a homeomorphism.*

Proof. Apply notation of Proposition 3. Propositions 1 and 3 imply that for each end point e of X the arc $ve \subset X$ is an inverse set under f, i.e., that $ve = f^{-1}(f(ve))$. Hence the partial mapping $f|ve$ is open [13, (7.2), p. 147] and, being monotone by Proposition 1, it is a homeomorphism. Therefore, to show that f is one-to-one it is enough to consider two points x_1 and x_2 of $X \setminus \{v\}$ with $x_1 \in ve_1$ and $x_2 \in ve_2$ for distinct end points e_1 and e_2 of X. Since $x_1x_2 = x_1v \cup vx_2$ and since the partial mapping $f|x_1x_2$ is monotone [4, Proposition 1, p. 307], we conclude that $f(x_1) \neq f(x_2)$ (because otherwise we would have $v' = f(v) \in f(x_1x_2) = \{f(x_1)\} = \{f(x_2)\}$ contrary to Proposition 3). The proof is complete. □

Observe that the mappings considered till now in Theorems 6, 8, 9, 10 and 11, under which a given nondegenerate continuum Y was the image of the Cantor fan (in particular, see conditions (i) of Theorems 6, 8 and 10), were confluent or retractions. Thus, according to (vi) of Theorem 6, the continua Y obtained in this way were smooth fans or arcs. Now we are going to consider light mappings of the Cantor fan. These mappings need not be confluent and, consequently, the continua Y being the images of F_C under these mappings need not be one-dimensional even. Moreover, since a perfect compact space X has dimension at most n if and only if X is the image of the Cantor set C under a mapping with preimages of points consisting of at most $n+1$ points [9, Section 45 II, p. 101], the cone over X is, in a natural way, the image of the cone F_C over C, and the induced mapping is obviously light.

Before we formulate our next result, which characterizes the images of the Cantor fan under light mappings, we recall a definition. A metric space X is said to be *uniformly pathwise connected* if there exists a family P of paths $p:[0, 1] \to X$ such that (i) for any two points x and y of X there exists a path $p \in P$ joining x and y; and (ii) for any $\varepsilon > 0$ there is a positive integer n such that each path p in P can be partitioned into n pieces of diameter at most ε, i.e., for each $p \in P$ there is a collection of $n+1$ numbers $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ such that $\operatorname{diam} p[t_{i-1}, t_i] \leq \varepsilon$ for each $i \in \{1, 2, \ldots, n\}$.

The most substantial part of the theorem below is the Kuperberg characterization of continuous images of the Cantor fan as uniformly pathwise connected continua [7, Theorem 3.5, p. 322]. We show that this class coincides with the images of F_C under light mappings.

12. Theorem. *The following conditions are equivalent for a nondegenerate continuum* Y:

(i) Y *is the image of the Cantor fan under a light mapping*;
(ii) Y *is the image of the Cantor fan under a mapping*;
(iii) Y *is uniformly pathwise connected.*

Proof. Since the equivalence between (ii) and (iii) is known [7, Theorem 3.5, p. 322], and the implication from (i) to (ii) is trivial, we close the proof showing that (ii) implies (i). So let a surjection $f: F_C \to Y$ be given. According to the Whyburn factorization theorem [13, Theorem 4.1, p. 141], f can be uniquely represented as the composition of two surjective mappings $f_1: F_C \to X$ and $f_2: X \to Y$ such that f_1 is monotone and f_2 is light. By Theorem 10 the continuum X is homeomorphic either to F_C or to F_C^{ω}. In the former case the proof is finished. In the latter case we apply the equivalence between conditions (vi) and (iii) of Theorem 6 to see that

X, being a smooth fan with the closed set $S(X)$, is the image of F_C under a light mapping g. Then the composition $f_2 g: F_C \to Y$ is a light surjection, which finishes the proof. $\square$

References

[1] J.J. Charatonik, Confluent mappings and unicoherence of continua, Fund. Math. 56 (1964) 213-220.
[2] J.J. Charatonik, On fans, Dissertationes Math. (Rozprawy Mat.) 54 (1967) 1-40.
[3] J.J. Charatonik and W.J. Charatonik, Fans with the property of Kelley, Topology Appl. 29 (1988) 73-78.
[4] J.J. Charatonik and C. Eberhart, On smooth dendroids, Fund. Math. 67 (1970) 297-322.
[5] C. Eberhart, A note on smooth fans, Colloq. Math. 20 (1969) 89-90.
[6] J.L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc. 52 (1942) 22-36.
[7] W. Kuperberg, Uniformly pathwise connected continua, Studies in Topology, in: Proceedings Conf. Univ. North Carolina, Charlotte, NC, 1974; (Math. Sect. Polish Acad. Sci., New York, 1975) 315-324.
[8] K. Kuratowski, Topology I (Academic Press, New York, 1966).
[9] K. Kuratowski, Topology II (Academic Press, New York, 1968).
[10] A. Lelek, On plane dendroids and their end points in the classical sense, Fund. Math. 49 (1961) 301-319.
[11] T. Mackowiak, Confluent mappings and smoothness of dendroids, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973) 719-725.
[12] R.W. Wardle, On a property of J.L. Kelley, Houston J. Math. 3 (1977) 291-299.
[13] G.T. Whyburn, Analytic Topology, Amer. Math. Soc. Colloquium Publ. 28 (Amer. Math. Soc., Providence, RI, 1942).